Homological algebra solutions Week 11

1. (a) It suffices to show that $d_{\text{cyl}}^{n+1} d_{\text{cyl}}^n = 0$ for any integer n. The matrix of this composition is given by

$$\begin{bmatrix} d_B^{n+1} & \mathrm{id}_{B^{n+2}} & 0 \\ 0 & -d_B^{n+2} & 0 \\ 0 & -f^{n+2} & d_C^{n+1} \end{bmatrix} \begin{bmatrix} d_B^n & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d_B^{n+1} & 0 \\ 0 & -f^{n+1} & d_C^n \end{bmatrix}$$
$$= \begin{bmatrix} d_B^{n+1} d_B^n & d_B^{n+1} - d_B^{n+1} & 0 \\ 0 & d_B^{n+2} d_B^{n+1} & 0 \\ 0 & f^{n+2} d_B^{n+1} - d_C^{n+1} f^{n+1} & d_C^{n+1} d_C^n \end{bmatrix}.$$

The matrix in the second line is 0 because B and C are cochain complexes and $f : B \to C$ is a morphism of cochain complexes.

(b) The chain maps $f, g : B \to C$ are chain homotopic if and only if there are maps $\{s^n : B^{n+1} \to C^n\}_{n \in \mathbb{Z}}$ such that

$$d_C^n s^n + s^{n+1} d_B^{n+1} = f^{n+1} - g^{n+1}$$

for all $n \in \mathbb{Z}$. Meanwhile, $(f, s, g) : \mathrm{cyl}(B) \to C$ is a morphism of cochain complexes iff

$$(f^{n+1}, s^{n+1}, g^{n+1}) d_{\text{cyl}}^n = d_C^n (f^n, s^n, g^n)$$
$$\begin{bmatrix} f^{n+1} & s^{n+1} & g^{n+1} \end{bmatrix} \begin{bmatrix} d_B^n & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d_B^{n+1} & 0 \\ 0 & -\mathrm{id}_{B^{n+1}} & d_B^n \end{bmatrix} = \begin{bmatrix} d_C^n \end{bmatrix} \begin{bmatrix} f^n & s^n & g^n \end{bmatrix}$$
$$\begin{bmatrix} f^{n+1} d_B^n \\ f^{n+1} - s^{n+1} d_B^{n+1} - g^{n+1} \\ g^{n+1} d_B^n \end{bmatrix}^T = \begin{bmatrix} d_C^n f^n \\ d_C^n s^n \\ d_C^n g^n \end{bmatrix}^T.$$

for all $n \in \mathbb{Z}$. The first and third components of this matrix equation hold because f and g are morphisms of cochain complexes. Thus, there exists a family of maps $\{s^n : B^{n+1} \to C^n\}_{n \in \mathbb{Z}}$ such that $(f, s, g) : \mathrm{cyl}(B) \to C$ is a morphism of cochain complexes iff there exists a family of maps $\{s^n : B^{n+1} \to C^n\}_{n \in \mathbb{Z}}$ such that

$$d_C^n s^n + s^{n+1} d_B^{n+1} = f^{n+1} - g^{n+1},$$

i.e., iff the maps $f, g : B \to C$ are chain homotopic.

(c) We have $\beta\alpha = \mathrm{id}_B$, thus to show that α is a chain homotopy equivalence, it suffices to show that $\alpha\beta$ is chain homotopic to $\mathrm{id}_{\mathrm{cyl}(B)}$. For each integer n, we define $s^n : \mathrm{cyl}(B)^{n+1} \to \mathrm{cyl}(B)^n$ by $s^n(b', b'', b) = (0, b, 0)$, i.e., s^n can be represented by the matrix

$$s^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{B^{n+1}} \\ 0 & 0 & 0 \end{bmatrix}.$$

We just have to confirm that $d^n_{\mathrm{cyl}(B)} s^n + s^{n+1} d^{n+1}_{\mathrm{cyl}(B)} = \alpha^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}}$ for any integer n. Since $\alpha\beta(b', b'', b) = \alpha(b' + b) = (b' + b, 0, 0)$, we can represent $\alpha^{n+1}\beta^{n+1}$ by the matrix

$$\alpha^{n+1}\beta^{n+1} = \begin{bmatrix} \mathrm{id}_{B^{n+1}} & 0 & \mathrm{id}_{B^{n+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\alpha^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}} = \begin{bmatrix} 0 & 0 & \mathrm{id}_{B^{n+1}} \\ 0 & -\mathrm{id}_{B^{n+2}} & 0 \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix}. \tag{1}$$

Also, we have

$$\begin{aligned} & d^n_{\mathrm{cyl}(B)} s^n + s^{n+1} d^{n+1}_{\mathrm{cyl}(B)} \\ &= \begin{bmatrix} d^n_B & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d^{n+1}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+1}} & d^n_B \end{bmatrix} s^n + s^{n+1} \begin{bmatrix} d^{n+1}_B & \mathrm{id}_{B^{n+2}} & 0 \\ 0 & -d^{n+2}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+2}} & d^{n+1}_B \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \mathrm{id}_{B^{n+1}} \\ 0 & 0 & -d^{n+1}_B \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{id}_{B^{n+2}} & d^{n+1}_B \\ 0 & 0 & 0 \end{bmatrix} \\ &= \alpha^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}}, \end{aligned}$$

by (1), as needed.

(d) As before, we have $\beta\alpha' = \mathrm{id}_B$, thus to conclude that α' is a chain homotopy equivalence, it suffices to show that $\alpha'\beta$ is chain homotpic to $\mathrm{id}_{\mathrm{cyl}(B)}$. Since $\mathbf{K}(A)$ is a category, we know that composition of morphisms is compatible with chain homotopy. Thus:

$$\alpha'\beta \sim \mathrm{id}_{\mathrm{cyl}(B)}\alpha'\beta \sim (\alpha\beta)\alpha'\beta \sim \alpha\beta \sim \mathrm{id}_{\mathrm{cyl}(B)},$$

as needed.

Now we wish to find maps $\{t^n : \mathrm{cyl}(B)^{n+1} \to \mathrm{cyl}(B)^n\}_{n\in\mathbb{Z}}$ such that

$$d^n_{\mathrm{cyl}(B)} t^n + t^{n+1} d^{n+1}_{\mathrm{cyl}(B)} = (\alpha')^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}}. \tag{2}$$

Define a chain map $\varphi : \mathrm{cyl}(B) \to \mathrm{cyl}(B)$ by $\varphi^n(x, y, z) = (-z, y, -x)$ for $x, z \in B^n$ and $y \in B^{n+1}$. This is in fact a chain map because

$$\begin{aligned} d^n_{\mathrm{cyl}(B)}\varphi^n &= \begin{bmatrix} d^n_B & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d^{n+1}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+1}} & d^n_B \end{bmatrix} \begin{bmatrix} 0 & 0 & -\mathrm{id}_{B^n} \\ 0 & \mathrm{id}_{B^{n+1}} & 0 \\ -\mathrm{id}_{B^n} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \mathrm{id}_{B^{n+1}} & -d^n_B \\ 0 & -d^{n+1}_B & 0 \\ -d^n_B & -\mathrm{id}_{B^{n+1}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -\mathrm{id}_{B^{n+1}} \\ 0 & \mathrm{id}_{B^{n+2}} & 0 \\ -\mathrm{id}_{B^{n+1}} & 0 & 0 \end{bmatrix} \begin{bmatrix} d^n_B & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d^{n+1}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+1}} & d^n_B \end{bmatrix} \\ &= \varphi^{n+1} d^n_{\mathrm{cyl}(B)}. \end{aligned}$$

We also note that (i) $\varphi\varphi = \mathrm{id}_{\mathrm{cyl}(B)}$, (ii) $\varphi\alpha = -\alpha'$, and (iii) $\beta\varphi = -\beta$. Recall that the maps $\{s^n : \mathrm{cyl}(B)^{n+1} \to \mathrm{cyl}(B)^n\}_{n\in\mathbb{Z}}$ from the previous subexercise of this problem satisfy for all $n \in \mathbb{Z}$:

$$\begin{aligned} d^n_{\mathrm{cyl}(B)} s^n + s^{n+1} d^{n+1}_{\mathrm{cyl}(B)} &= \alpha^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}} \\ d^n_{\mathrm{cyl}(B)}(\varphi^n s^n \varphi^{n+1}) + (\varphi^{n+1} s^{n+1} \varphi^{n+2}) d^{n+1}_{\mathrm{cyl}(B)} &= (\alpha')^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}}. \end{aligned}$$

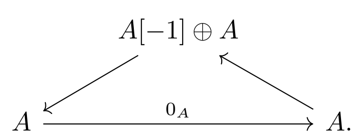
Thus if for each integer n we define $t^n : \mathrm{cyl}(B)^{n+1} \to \mathrm{cyl}(B)^n$ by $t^n = \varphi^n s^n \varphi^{n+1}$, then the desired equation (2) is satisfied. Specifically, for $x, z \in B^{n+1}$ and $y \in B^{n+2}$, we have

$$t^n(x, y, z) = \varphi^n s^n(-z, y, -x) = \varphi^n(0, -x, 0) = (0, -x, 0).$$

2. (a) We first determine the mapping cone $\mathrm{cone}(0_A)$ of $0_A : A \to A$. The object in degee n of the complex $\mathrm{cone}(0_A)$ is $A^{n+1} \oplus A^n$, and the differential $d^n : \mathrm{cone}(0_A)^n \to \mathrm{cone}(0_A)^{n+1}$ is defined to be

$$d^n = (-d^{n+1}_A, d^n_A). \tag{3}$$

Recall that the translate $A[-1]$ has nth object $A[-1]^n = A^{n+1}$ and nth differential $-d^{n+1}_A$. Thus $\mathrm{cone}(0_A)$ is precisely the cochain complex $A[-1] \oplus A$. It follows that we have a strict (and therefore exact) triangle



Next, we find an exact triangle for the identity morphism $1_A : A \to A$. The mapping cone $\mathrm{cone}(1_A)$ of the identity on A is by definition the cone $\mathrm{cone}(A)$ of the complex A. In Week 4 (cf. Proposition 3.9 of Week 4's lecture notes), we saw that $\mathrm{cone}(A)$ is split exact and therefore contractible.

Hence there is a homotopy equivalence $h : \operatorname{cone}(1_A) \to 0^\bullet$, where $0^\bullet$ is the zero cochain complex. Thus we have a diagram

$$\begin{array}{ccccccc} A & \xrightarrow{1_A} & A & \longrightarrow & 0^\bullet & \longrightarrow & A[-1] \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle h} & & \downarrow \\ A & \xrightarrow{1_A} & A & \longrightarrow & \operatorname{cone}(1_A) & \longrightarrow & A[-1], \end{array}$$

in which the vertical maps are homotopy equivalences and each square commutes up to homotopy. We conclude that the triangle

$$\begin{array}{ccc} & 0^\bullet & \\ \swarrow & & \nwarrow \\ A & \xrightarrow{\quad 1_A \quad} & A \end{array}$$

is exact.

(b, i) We first assume that the triangle (u, v, w) on A, B, C is the strict triangle on $u : A \to B$, i.e., $C = \operatorname{cone}(u)$ and $v : B \to \operatorname{cone}(u)$ and $w : \operatorname{cone}(u) \to A[-1]$ are the usual maps. We want to find a homotopy equivalence $\beta : A[-1] \to \operatorname{cone}(v)$ such that we have a diagram

$$\begin{array}{ccccccc} B & \xrightarrow{v} & C & \xrightarrow{w} & A[-1] & \xrightarrow{-u[-1]} & B[-1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1} \\ B & \xrightarrow{v} & C & \xrightarrow{w'} & \operatorname{cone}(v) & \xrightarrow{\delta} & B[-1] \end{array} \tag{4}$$

that commutes up to homotopy, where the bottom row is the strict triangle on $v : B \to C$. We first clarify what the complex $\operatorname{cone}(v)$ is. Its nth object is

$$\operatorname{cone}(v)^n = B^{n+1} \oplus C^n = B^{n+1} \oplus \operatorname{cone}(u)^n = B^{n+1} \oplus A^{n+1} \oplus B^n,$$

and its nth differential is

$$d^n = \begin{bmatrix} -d_B^{n+1} & 0 \\ -v^{n+1} & d^n_{\operatorname{cone}(u)} \end{bmatrix} = \begin{bmatrix} -d_B^{n+1} & 0 & 0 \\ 0 & -d_A^{n+1} & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d_B^n \end{bmatrix}.$$

Define $\beta : A[-1] \to \operatorname{cone}(v)$ by letting $\beta^n : A^{n+1} \to \operatorname{cone}(v)^n$ be the morphism $\beta^n = (-u^{n+1}, \mathrm{id}_{A^{n+1}}, 0)$ for each $n \in \mathbb{Z}$. This is a chain map because for any n:

$$\begin{aligned}
\beta^{n+1} d^n_{A[-1]} &= -\beta^{n+1} d^{n+1}_A \\
&= \begin{bmatrix} u^{n+2} d^{n+1}_A \\ -d^{n+1}_A \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} d^{n+1}_B u^{n+1} \\ -d^{n+1}_A \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -d^{n+1}_B & 0 & 0 \\ 0 & -d^{n+1}_A & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d^n_B \end{bmatrix} \begin{bmatrix} -u^{n+1} \\ \mathrm{id}_{A^{n+1}} \\ 0 \end{bmatrix} \\
&= d^n_{\mathrm{cone}(v)} \beta^n.
\end{aligned}$$

Next, define $\gamma : \mathrm{cone}(v) \to A[-1]$ by letting $\gamma^n : \mathrm{cone}(v)^n \to A^{n+1}$ be the morphism $\gamma^n = (0, \mathrm{id}_{A^{n+1}}, 0)$. This is a morphism of chain complexes because

$$\begin{aligned}
d^n_{A[-1]} \gamma^n &= (0, -d^{n+1}_A, 0) \\
&= (0, \mathrm{id}_{A^{n+2}}, 0) \begin{bmatrix} -d^{n+1}_B & 0 & 0 \\ 0 & -d^{n+1}_A & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d^n_B \end{bmatrix} \\
&= \gamma^{n+1} d^n_{\mathrm{cone}(v)}.
\end{aligned}$$

Moreover, we have $\gamma\beta = \mathrm{id}_{A[-1]}$, thus to show that $\beta : A[-1] \to \mathrm{cone}(v)$ is a homotopy equivalence, it suffices to show that $\beta\gamma$ is homotopic to the identity on $\mathrm{cone}(v)$. For each integer n, define $s^n : \mathrm{cone}(v)^{n+1} \to \mathrm{cone}(v)^n$ by $(x, y, z) \mapsto (z, 0, 0)$. Then:

$$\begin{aligned}
& d^n_{\mathrm{cone}(v)} s^n + s^{n+1} d^{n+1}_{\mathrm{cone}(v)} = \\
& \begin{bmatrix} -d^{n+1}_B & 0 & 0 \\ 0 & -d^{n+1}_A & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d^n_B \end{bmatrix} \begin{bmatrix} 0 & 0 & \mathrm{id}_{B^{n+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & \mathrm{id}_{B^{n+2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -d^{n+2}_B & 0 & 0 \\ 0 & -d^{n+2}_A & 0 \\ -\mathrm{id}_{B^{n+2}} & -u^{n+2} & d^{n+1}_B \end{bmatrix} \\
& = \begin{bmatrix} 0 & 0 & -d^{n+1}_B \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix} + \begin{bmatrix} -\mathrm{id}_{B^{n+2}} & -u^{n+2} & d^{n+1}_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
& = \begin{bmatrix} -\mathrm{id}_{B^{n+2}} & -u^{n+2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix}.
\end{aligned} \tag{5}$$

On the other hand, we have $\beta^n\gamma^n(x,y,z) = \beta^n(y) = (-u^{n+1}(y), y, 0)$, thus:

$$\begin{aligned}
&\beta^{n+1}\gamma^{n+1} - \mathrm{id}_{\mathrm{cone}(v)^{n+1}} \\
&= \begin{bmatrix} 0 & -u^{n+2} & 0 \\ 0 & +\mathrm{id}_{A^{n+2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathrm{id}_{B^{n+2}} & 0 & 0 \\ 0 & \mathrm{id}_{A^{n+2}} & 0 \\ 0 & 0 & \mathrm{id}_{B^{n+1}} \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{id}_{B^{n+2}} & -u^{n+2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix} \\
&= d^n_{\mathrm{cone}(v)} s^n + s^{n+1} d^{n+1}_{\mathrm{cone}(v)},
\end{aligned}$$

by (5), and we conclude that $\beta\gamma$ is homotopic to $\mathrm{id}_{\mathrm{cone}(v)}$. Thus $\beta : A[-1] \to \mathrm{cone}(v)$ is a homotopy equivalence. It remains to show that the squares in (4) commute up to homotopy. We first note that for $x \in A[-1]^n = A^{n+1}$, we have

$$\delta^n\beta^n(x) = \delta^n(-u^{n+1}(x), x, 0) = -u^{n+1}(x) = -u[-1]^n(x),$$

thus $\delta\beta = -u[-1]$, and the right square in (4) commutes. Since the left square commutes trivially, it remains to check that the central square commutes up to homotopy, i.e., $\beta w \sim w'$. Recalling that $w : \mathrm{cone}(u) \to A[-1]$ is the usual map from the mapping cone of $u : A \to B$ to $A[-1]$, we find $\gamma w' = w$, thus $\gamma w' \sim w$. But γ is a homotopy equivalence and its inverse is β, thus $w' \sim \beta w$, as needed. This completes the proof that

$$\begin{array}{ccccc} & & A[-1] & & \\ & \swarrow^{-u[-1]} & & \nwarrow^{w} & \\ B & & \xrightarrow{\quad v \quad} & & C \end{array} \tag{6}$$

is exact when (u, v, w) is a strict triangle on A, B, C. Since an exact triangle is by definition isomorphic to a strict triangle, the rotate in (6) remains exact when (u, v, w) is an exact triangle on A, B, C.

(b, ii) Once again, we assume that the triangle (u, v, w) on A, B, C is the strict triangle on $u : A \to B$, i.e., C is the mapping cone of u. We let δ be the map $-w[1] : C[1] \to A$, and want to find a homotopy equivalence $\sigma : B \to \mathrm{cone}(\delta)$ such that the diagram

$$\begin{array}{ccccccc} C[1] & \xrightarrow{\delta} & A & \xrightarrow{u} & B & \xrightarrow{v} & C \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle 1} \\ C[1] & \xrightarrow{\delta} & A & \xrightarrow{u'} & \mathrm{cone}(\delta) & \xrightarrow{v'} & C \end{array}$$

commutes up to homotopy. We first demonstrate that $\mathrm{cone}(\delta) = \mathrm{cyl}(u)$. For any integer n, we have

$$\mathrm{cone}(\delta)^n = C[1]^{n+1} \oplus A^n = C^n \oplus A^n = A^{n+1} \oplus B^n \oplus A^n \cong \mathrm{cyl}(u)^n.$$

Moreover, the differential $d^n_{\mathrm{cone}(\delta)} : \mathrm{cone}(\delta)^n \to \mathrm{cone}(\delta)^{n+1}$ is

$$d^n_{\mathrm{cone}(\delta)} = \begin{bmatrix} -d^{n+1}_{C[1]} & 0 \\ -\delta^{n+1} & d^n_A \end{bmatrix} = \begin{bmatrix} d^n_C & 0 \\ w^n & d^n_A \end{bmatrix} = \begin{bmatrix} -d^{n+1}_A & 0 & 0 \\ -u^{n+1} & d^n_B & 0 \\ \mathrm{id}_{A^{n+1}} & 0 & d^n_A \end{bmatrix}.$$

It is straightforward to verify that the maps $\vartheta^k : \mathrm{cone}(\delta)^k \to \mathrm{cyl}(u)^k$ given by $\vartheta^k(x, y, z) = (z, x, y)$ define an isomorphism $\vartheta : \mathrm{cone}(\delta) \to \mathrm{cyl}(u)$ of chain complexes. Our task thus becomes to find a homotopy equivalence $\tilde{\sigma} : B \to \mathrm{cyl}(u)$ such that the diagram

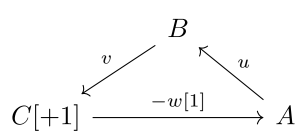
$$\begin{array}{ccccccc} C[1] & \xrightarrow{\delta} & A & \xrightarrow{u} & B & \xrightarrow{v} & C \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \tilde{\sigma}} & & \downarrow{\scriptstyle 1} \\ C[1] & \xrightarrow{\delta} & A & \xrightarrow{\tilde{u}} & \mathrm{cyl}(u) & \xrightarrow{\tilde{v}} & C \end{array} \tag{7}$$

is commutative up to homotopy, where $\tilde{u} : A \to \mathrm{cyl}(U)$ is the inclusion of A in $\mathrm{cyl}(u)$ and $\tilde{v} : \mathrm{cyl}(u) \to C$ is the projection.

Before continuing, we note that subexercise (d) of Problem 1 holds in a more general context (cf. Exercise 1.5.4 in Weibel): if $u : A \to B$ is a morphism of chain complexes, and we define a map $\alpha' : B \to \mathrm{cyl}(u)$ by $\alpha'(b) = (0, 0, b)$ and a map $\beta : \mathrm{cyl}(u) \to B$ by $\beta(a', a, b) = f(a') + b$, then α' is a homotopy equivalence with inverse β. In particular, we may define $\tilde{\sigma} : B \to \mathrm{cyl}(u)$ to be the map α' to obtain a homotopy equivalence such that the diagram in (7) commutes up to homotopy. Indeed, we have $v = \tilde{v}\tilde{\sigma}$, so the right square in (7) commutes. As for the central square, we have $\beta\tilde{u} = u$, thus

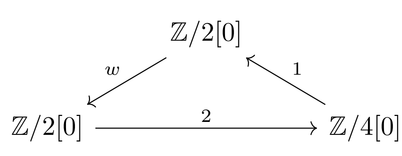
$$\beta\tilde{u} \sim u \implies \tilde{u} \sim \tilde{\sigma}u,$$

since $\tilde{\sigma}$ is the inverse of the homotopy equivalence β. This completes the proof that given a strict triangle (u, v, w) on A, B, C, the rotate



is an exact triangle, and by the same reasoning as before, we conclude that this remains true when (u, v, w) is only an exact triangle on A, B, C.

3. We let $\mathbb{Z}/2[0]$ and $\mathbb{Z}/4[0]$ be the cochain complexes concentrated in degree 0. The claim is that there is no morphism $w : \mathbb{Z}/2[0] \to \mathbb{Z}/2[-1]$ such that



is an exact triangle. We suppose toward a contradiction that there is such a w.

Let $\varphi : \mathbb{Z}/2[0] \to \mathbb{Z}/4[0]$ be the extension of the map $\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4$. Then the exactness of $(\varphi, 1, w)$ on $\mathbb{Z}/2[0], \mathbb{Z}/4[0], \mathbb{Z}/2[0]$, the exactness of the strict triangle on $\varphi : \mathbb{Z}/2[0] \to \mathbb{Z}/4[0]$, and the the TR3 axiom of the triangulated category $\mathbf{K}(A)$ yield a morphism $\psi : \mathbb{Z}/2[0] \to \operatorname{cone}(\psi)$ such that the following diagram commutes

$$\begin{array}{ccccccc} \mathbb{Z}/2[0] & \xrightarrow{\varphi} & \mathbb{Z}/4[0] & \xrightarrow{1} & \mathbb{Z}/2[0] & \xrightarrow{w} & \mathbb{Z}/2[-1] \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \exists\psi} & & \downarrow{\scriptstyle 1} \\ \mathbb{Z}/2[0] & \xrightarrow{\varphi} & \mathbb{Z}/4[0] & \longrightarrow & \operatorname{cone}(\varphi) & \longrightarrow & \mathbb{Z}/2[-1]. \end{array}$$

The 5-lemma for exact triangles tells us that ψ is in fact a homotopy equivalence. Thus we have a quasi-isomorphism between $\mathbb{Z}/2[0]$ and $\operatorname{cone}(\varphi)$. But $H^{-1}(\mathbb{Z}/2[0]) = \{0\}$, whereas

$$H^{-1}(\operatorname{cone}(\varphi)) = \frac{\ker(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4)}{\operatorname{im}(0 \to \mathbb{Z}/2)} = \ker(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4) \neq \{0\}.$$

This contradiction tells us that there is no morphism w such that $(2, 1, w)$ is an exact triangle on $(\mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/2)$.

4. Let $\mathcal{D}$ be a triangulated category, and suppose we have a diagram

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ & & \downarrow{\scriptstyle g} & & & & \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA' \end{array},$$

where the rows are exact triangles. We assume that $v'gu = 0$, we want to show that there are maps $f : A \to A'$ and $h : C \to C'$ which assemble with g to get a map of exact triangles, i.e. fit into the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA' \end{array}.$$

To do this, recall that for all $X \in \mathcal{D}$, we have that $\operatorname{Hom}(X, -) : \mathcal{D} \to \mathrm{Ab}$ is a cohomological functor. In particular the following is an exact sequence of abelian groups

$$\operatorname{Hom}(A, A') \xrightarrow{(u')^*} \operatorname{Hom}(A, B') \xrightarrow{(v')^*} \operatorname{Hom}(A, C') \xrightarrow{(w')^*} \operatorname{Hom}(A, TA').$$

The assumptions imply that $gu \in \operatorname{Hom}(A, B')$ is mapped to 0 in $\operatorname{Hom}(A, C')$. By exactness this gives a map $f : A \to A'$ such that post-composing with

u' yields the composition gu. Now axiom (TR3) gives us a map $h : C \to C'$ such that

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA' \end{array}$$

commutes, which is what we wanted to show. This concludes the proof of this exercise.

5. Let $\mathcal{A}$ be an abelian category, and consider the category of graded $\mathcal{A}$-objects viewed as the functor category $\mathcal{A}^{\mathbb{Z}}$, where $\mathbb{Z}$ is the set $\mathbb{Z}$ viewed as a discrete category. There is an obvious automorphism $\mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by precomposing a functor $\mathbb{Z} \to \mathcal{A}$ by the the "+1" map, so that $T(A_\bullet)_n = A_{n-1}$ for some $\mathbb{Z}$-graded object in $\mathcal{A}$. We call a triangle $(A_\bullet, B_\bullet, C_\bullet, u, v, w)$ exact if for all n the sequence

$$A_n \xrightarrow{u} B_n \xrightarrow{v} C_n \xrightarrow{w} A_{n-1}$$

is exact in A.

Consider the case $\mathcal{A} = \text{Ab}$, we claim that in this case the category of graded objects with the degree shift automorphisms satisfies (TR1) and (TR2), but not (TR3). The fact that satisfies (TR2) is immediate, and as such we omit the details. Now suppose we have a morphism $A_\bullet \xrightarrow{u_\bullet} B_\bullet$, we want to fit it in an exact triangle. For this, the graded object $C_n = \text{coker}(u_n) \oplus A_{n-1}$ will do the job. Indeed, for the map $B_\bullet \to C_\bullet$ in degree n take the composite $B_n \to \text{coker}(u_n) \to \text{coker}(u_n) \oplus A_{n-1}$, and for the map $C_\bullet \to A_{\bullet-1}$ in degree n take the obvious projection map. It is now clear from the fact that exactness in a functor category can be checked objectwise that the image of $B_\bullet \to C_\bullet$ is the kernel of $C_\bullet \to A_{\bullet-1}$.

We now show that this category doesn't satisfy (TR3). Notice that exact triangles $(A_\bullet, B_\bullet, C_\bullet)$ with $A_\bullet$ concentrated in degree -1 and both $B_\bullet$ and $C_\bullet$ concentrated in degree 0 correspond to short exact sequences of abelian groups in an obvious way, and under this correspondence, if our category $\mathcal{A}^{\mathbb{Z}}$ were to satisfy (TR3) then any partial map of short exact sequences of abelian group

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_0 & \longrightarrow & C_0 & \longrightarrow & A_{-1} & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & B'_0 & \longrightarrow & C'_0 & \longrightarrow & A'_{-1} & \longrightarrow & 0 \end{array},$$

could be completed into a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_0 & \longrightarrow & C_0 & \longrightarrow & A_{-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B'_0 & \longrightarrow & C'_0 & \longrightarrow & A'_{-1} & \longrightarrow & 0 \end{array}.$$

This would in particular enable us to obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array},$$

which would by the 5 lemma imply that $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is obviously false, thus yielding the desired contradiction.
We will now show that if instead $\mathcal{A}$ is the category of vector spaces over a field, then the category of graded objects with the shift automorphism is in fact a triangulated category. The fact that axiom (TR1) and (TR2) are satisfied follow by a reasoning perfectly analogous to the abelian group case. Now let us show that $(k - \mathrm{Vect})^{\mathbb{Z}}$ satisfies (TR3). Assume we have a partial map of exact triangles

$$\begin{array}{ccccccc} U_\bullet & \longrightarrow & V_\bullet & \longrightarrow & W_\bullet & \longrightarrow & U_{\bullet-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ U'_\bullet & \longrightarrow & V'_\bullet & \longrightarrow & W'_\bullet & \longrightarrow & U'_{\bullet-1} \end{array}.$$

We need to find a map $W_\bullet \to W'_\bullet$ making the above diagram commute. This is equivalent to defining maps w_n for all $n \in \mathbb{Z}$ fitting in the following commutative diagrams with exact rows

$$\begin{array}{ccccccc} U_n & \longrightarrow & V_n & \longrightarrow & W_n & \longrightarrow & U_{n-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ U'_n & \longrightarrow & V'_n & \longrightarrow & W'_n & \longrightarrow & U'_{n-1} \end{array}.$$

The fact that there exists a map $w_n : W_n \to W'_n$ making the above diagram commutes follows from basic linear algebra.
Finally, we show that this category satisfies the (TR4) axiom. To do this we introduce a different perspective on this axiom than the one seen in class, coming from the stacks project [**stacks-project**]. We want to insist on the fact that this is nothing more than a shift in perspective.
Suppose $\mathcal{D}$ is a category with an automorphism $T : \mathcal{D} \to FD$ and collection of distinguished triangles. We say that it satisfies (TR4) if given two composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and distinguished triangles (A, B, Q_1, f, p_1, d_1), $(A, C, Q_2, g \circ f, p_2, d_2)$ and (B, C, Q_3, g, p_3, d_3), then there exists a fourth distinguished triangle $(Q_1, Q_2, Q_3, a, b, T(p_1) \circ d_3)$. And furthermore we require that the triple

$$(\mathrm{Id}_X, g, a) : (X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$$

is a morphism of triangles and

$$(f, \mathrm{Id}_Z, b) : (X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$$

as well. When using (TR4), we will allow ourselves to only refer to the morphism f, g as input, leaving the rest of the input as implicit.
We now prove that the category of graded vector spaces with the degree shift automorphism satisfies (TR4). First notice that by the (standard) 5-lemma and (TR2), up to isomorphism any exact triangle $(V_\bullet, V'_\bullet, V''_\bullet, a, b, c)$ of graded vector spaces can be identified with $(V_\bullet, V'_\bullet, \operatorname{coker}(a_\bullet) \oplus V_{\bullet-1}, a, \iota \circ q, \pi)$, where $\iota : \operatorname{coker}(a_\bullet) \to \operatorname{coker}(a_\bullet) \oplus V_{\bullet-1}$ is the obvious inclusion, $q : B \to \operatorname{coker}(a_\bullet)$ is the canonical map and $\pi : \operatorname{coker}(a_\bullet) \oplus V_{\bullet-1} \to V_{\bullet-1}$ is the obvious projection. With this in hand, we see that we may start with a diagram of the form

$$\begin{array}{ccccccc}
U_\bullet & \xrightarrow{f} & V_\bullet & \longrightarrow & V_\bullet/U_\bullet \oplus U_{\bullet-1} & \longrightarrow & U_{\bullet-1} \\
& \searrow^{g\circ f} & & \searrow^{g} & & & \\
& & & W_\bullet & \longrightarrow W_\bullet/U_\bullet \oplus U_{\bullet-1} & \longrightarrow & U_{\bullet-1} \\
& & & & \searrow & & \\
& & & & W_\bullet/V_\bullet \oplus V_{\bullet-1} & \longrightarrow & V_{\bullet-1}
\end{array}$$

where we denote the cokernel of a map by the quotient of the codomain by the domain and where the sequences of composable moprhisms which look like exact triangles are exact triangles. If we construct an exact triangle on the second to last column, along with some moprhisms of triangles, we will be done. This will be given by the following triangle, where we will detail the definitions of the morphisms but will leave the verification of exactness to the reader

$$V_\bullet/U_\bullet \oplus U_{\bullet-1} \xrightarrow{\tilde{g}\oplus \mathrm{Id}} W_\bullet/U_\bullet \oplus U_{\bullet-1} \xrightarrow{q\oplus f} W_\bullet/V_\bullet \oplus V_{\bullet-1} \xrightarrow{q\circ\pi} V_{\bullet-1}/U_{\bullet-1} \oplus U_{\bullet-2}.$$

So as to not overclutter the notation, we have omitted placeholder subscripts in our notation. We now detail all the maps: the map $\tilde{g}$ is the map induced by g on the quotients; the map $q : W_\bullet/U_\bullet \to W_\bullet/V_\bullet$ is the obvious quotient map; the map $\pi : W_\bullet/V_\bullet \oplus V_{\bullet-1} \to V_{\bullet-1}$ is the obvious projection map; and $q : V_{\bullet-1} \to V_{\bullet-1}/U_{\bullet-1}$ is the obvious quotient map. As we stated above, we leave the verification that all of these maps make $(V_\bullet/U_\bullet \oplus U_{\bullet-1}, W_\bullet/U_\bullet \oplus U_{\bullet-1}, W_\bullet/V_\bullet \oplus V_{\bullet-1})$ an exact triangle.
It only remains to verify that

$$(\mathrm{Id}, g, \tilde{g} \oplus \mathrm{Id}) : (U_\bullet, V_\bullet, V_\bullet/U_\bullet \oplus U_{\bullet-1} \to (U_\bullet, W_\bullet, W_\bullet/U_\bullet \oplus U_{\bullet-1})$$

and

$$(f, \mathrm{Id}, q \oplus f) : (U_\bullet, W_\bullet, W_\bullet/U_\bullet \oplus U_{\bullet-1}) \to (V_\bullet, W_\bullet, W_\bullet/V_\bullet \oplus V_{\bullet-1})$$

are moprhisms of exact triangles (we have allowed ourselves the abuse of notation of only referring to the objects of the exact triangles). But of these are completely obvious from our choice of maps.

6. Let $\mathcal{D}$ be a triangulated category and consider a commutative square in this category

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle u}\downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} .$$

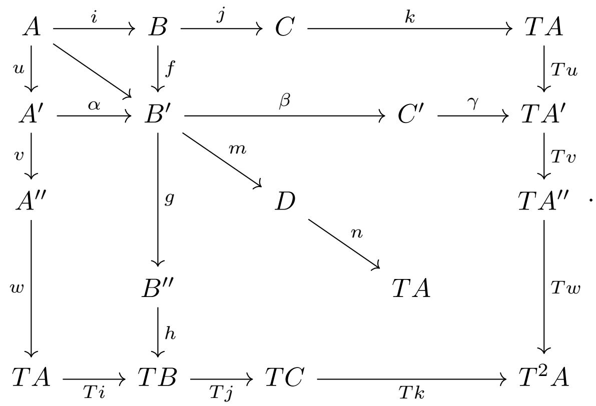
We want to show that we can always extend such a square to a diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & TA \\ {\scriptstyle u}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tu} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \\ {\scriptstyle v}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tv} \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & TA'' \\ {\scriptstyle w}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tw} \\ TA & \xrightarrow[Ti]{} & TB & \xrightarrow[Tj]{} & TC & \xrightarrow[Tk]{} & T^2A \end{array}$$

where all the rows and columns are exact triangles and all the squares commute except the bottom right one which anti-commutes. We will allow ourselves the abuse notation of suppressing the morphisms when specifying a triangle. First, we use four applications of (TR1) to extend our square to the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & TA \\ {\scriptstyle u}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle Tu} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \\ {\scriptstyle v}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle Tv} \\ A'' & & B'' & & & & TA'' \\ {\scriptstyle w}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle Tw} \\ TA & \xrightarrow[Ti]{} & TB & \xrightarrow[Tj]{} & TC & \xrightarrow[Tk]{} & T^2A \end{array} .$$

The commutativity of this diagram follows from the fact that the composite of two morphism of an exact triangle are 0. To conclude, we want to use axiom (TR4), and to do this, we add a temporary object, which will be useful in constructing the remaining morphisms, but won't appear in our final diagram. This object is obtained by applying axiom (TR1) to the map $A \to B'$, yielding the following diagram (where we have also added names to all of our maps for convenience):



We will use the variant on axiom (TR4) mentionned in the previous exercise, which we do not recall here. First we have two applications of (TR4). One on the composition $A \xrightarrow{i} B \xrightarrow{f} B'$ and one on the composition $A \xrightarrow{u} A' \xrightarrow{\alpha} B'$. The first of these gives us an exact triangle $(C, D, B'', t_1, s_1, T(j) \circ h)$ along with the maps of triangles

$$(\mathrm{Id}, f, t_1) : (A, B, C) \to (A, B', D)$$

and

$$(i, \mathrm{Id}, s_1) : (A, B', D) \to (B, B', B'').$$

The second application of (TR4) gives us an exact triange $(A'', D, C', t_2, s_2, Tv \circ \gamma)$ along with maps of triangles

$$(\mathrm{Id}, \alpha, t_2) : (A, A', A'') \to (A, B', D)$$

and

$$(u, \mathrm{Id}, s_2) : (A, B', D) \to (A', B', C').$$

Now considering the composite $s_1 \circ t_2 : A'' \to B''$, and applying (TR1) to get an exact triangle $(A'', B'', C'', s_1 \circ t_2, \sigma, \tau)$ and (TR2) on $(C, D, B'', t_1, s_1, T(j) \circ h)$ to get an exact triangle $(D, B'', TC, s_1, T(j) \circ h, -Tt_1)$, we have the necessary set up to apply (TR4) to the composition $s_1 \circ t_2$. This gives us an exact triangle $(C', C'', TC, p, q, Ts_2 \circ (-Tt_1))$, and two morphisms of triangles

$$(\mathrm{Id}, s_1, p) : (A'', D, C') \to (A'', B'', C'')$$

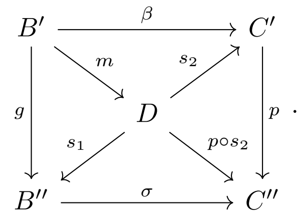
and

$$(t_2, \mathrm{Id}, q) : (A'', B'', C'') \to (D, B'', TC).$$

We can rotate the triangle $(C', C'', TC, p, q, Ts_2 \circ (-Tt_1))$, to obtain the triangle $(C, C', C'', s_2 \circ t_1, p, q)$. We can use all of this to construct the following diagram, whose commutativity will occupy us for the rest of this exercise

$$\begin{array}{ccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & TA \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle s_2\circ t_1} & & \downarrow{\scriptstyle Tu} \\
A' & \xrightarrow{\alpha} & B' & \xrightarrow{\beta} & C' & \xrightarrow{\gamma} & TA' \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle Tv} \\
A'' & \xrightarrow{s_1\circ t_2} & B'' & \xrightarrow{\sigma} & C'' & \xrightarrow{\tau} & TA'' \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle Tw} \\
TA & \xrightarrow{Ti} & TB & \xrightarrow{Tj} & TC & \xrightarrow{Tk} & T^2A
\end{array} .$$

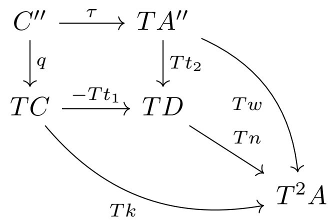
In what follows we will refer to triangles only by their objects. Commutativity of the three squares of the first column follows from composing the morphisms of triangles $(A, A', A'') \to (A, B', D)$ and $(A, B', D) \to (B, B', B'')$. Similarly, the three sqaures of the top row commute by composing the morphisms of triangles $(A, B, C) \to (A, B', D)$ and $(A, B', D) \to (A', B', C')$. The middle square commutes, by adding D in the middle, and the fact that in the following diagram all 4 triangles commute, as can be seen by inspecting the various morphisms of triangles (TR4) has granted us



The bottom middle square commute thanks to the morphism of triangles $(A'', B'', C'') \to (D, B'', TC)$ and because the map $B'' \to TC$ is $T(j) \circ h$. Similarly, the middle right square commutes thanks to the map of triangles $(A'', D, C') \to (A'', B'', C'')$ and the fact that the map $C' \to TA''$ factors as $C' \to TA \to TA''$. For anticommutativity of the bottom right diagram, because $(A'', B'', C'') \to (D, B'', TC)$ is a morphism of triangles, we have that the following square commutes

$$\begin{array}{ccc}
C'' & \xrightarrow{\tau} & TA'' \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle Tt_2} \\
TC & \xrightarrow{-Tt_1} & TD
\end{array} .$$

We can now post compose by $Tn : TD \to T^2A$, which because $(A, A', A'') \to (A, B', D)$ and $(A, B, C) \to (A, B', D)$ are morphisms of triangles, gives us a diagram



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which proves the deisred anticomutativity. This concludes the exercise.